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THE PROPAGATION OF WAVES FROM A SPHERICAL CAVITY IN AN ACOUSTIC HALF-SPACE*

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The propagation of a non-stationary acoustic pressure wave from a spherical cavity situated in a half-space filled with an ideal compressible fluid is considered. An infinite system of algebraic equations is obtained in Laplace transformation image space, the solution of which is found in the form of a series in exponential functions. An infinite system of Volterra integral equations of the second kind in the space of the originals was obtained in /1/ of this problem, and was solved numerically.

1. Formulation of the problem. Suppose the centre O of a spherical cavity of radius R $(h > R)$, in which the origin of a spherical system of coordinates r, θ, ϑ is placed, is situated at a distance h from a free boundary $z = 0$ in an acoustic half-space $z \geqslant 0$. A pressure p_1 is applied to the surface of the cavity. The velocity potential of the acoustic medium φ, taking into account the axial symmetry of the problem, satisfies the wave equation and the corresponding boundary conditions at the free boundary and on the surface of the cavity

$$\frac{\partial^2\varphi}{\partial\tau^2} = \Delta\varphi; \quad \frac{\partial\varphi}{\partial\tau}\Big|_{z=0} = 0, \quad \frac{\partial\varphi}{\partial\tau}\Big|_{r=1} = -p_1(\tau, \theta) \tag{1.1}$$

At the initial instant of time $\tau = 0$ the medium is in a state of rest

$$\varphi|_{\tau=0} = \partial\varphi/\partial\tau|_{\tau=0} = 0 \tag{1.2}$$

We will use the following dimensionless quantities

$$r = \frac{r'}{R}, \quad h = \frac{h'}{R}, \quad p = \frac{p'}{\rho c^2}, \quad \tau = \frac{ct}{R}, \quad \varphi = \frac{\varphi'}{cR}$$

where the prime denotes dimensional quantities, t is the time, and ρ and c are the density and velocity of propagation of sound in the acoustic medium.

2. Method of solution. We will apply an integral Laplace transformation with respect to time τ to the initial-boundary value problem (1.1), (1.2) (L is the Laplace transformant and s is the transformation parameter). In transformation space the velocity potential can be represented in the form

$$\varphi^L = \sum_{n=0}^{\infty}\left[\frac{1}{\sqrt{r}} A_n^L(s) K_{n+1/2}(sr) P_n(\cos\theta) + \frac{1}{\sqrt{r_1}} B_n^L(s) K_{n+1/2}(sr_1) P_n(\cos\theta_1)\right] \tag{2.1}$$

where $K_{n+1/2}(x)$ is the modified Bessel function of the second kind, $P_n(x)$ are Legendre polynomials, $A_n^L(s)$ and $B_n^L(s)$ are unknown functions, which are found from the boundary conditions, and r_1, θ_1 are the coordinates of a spherical system of coordinates with centre at the point O_1, symmetrical about the point O relative to the boundary of the half-space $z = 0$ (see the figure).

Taking into account the relations

$$\begin{gathered} r|_{z=0} = r_1|_{z=0}, \quad \theta|_{z=0} + \theta_1|_{z=0} = \pi \\ (-1)^n P_n(\cos\theta)|_{z=0} = P_n(\cos\theta_1)|_{z=0} \end{gathered} \tag{2.2}$$

and satisfying the first boundary condition in (1.1), we obtain the relation

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$$B_n^L(s) = (-1)^{n+1} A_n^L(s) \tag{2.3}$$

Using the summation theorem for the functions $K_{n+1/2}(x)$ /2/, we can represent the potential φ^L as follows:

$$\varphi^L = \sum_{n=0}^{\infty} \frac{1}{\sqrt{r}} \Big[A_n^L(s) K_{n+1/2}(sr) + (-1)^n (2n+1) I_{n+1/2}(sr) \times \tag{2.4}$$
$$\sum_{p=0}^{\infty} (-1)^{p+1} A_p^L(s) \sum_{\sigma=|p-n|}^{p+n} b_\sigma^{(n0p0)} \sqrt{\frac{\pi}{4hs}} K_{\sigma+1/2}(2hs) \Big] P_n(\cos\theta)$$

where $b_\sigma^{(n0p0)}$ are the Clebsch-Gordon coefficients /2/, and $I_{n+1/2}(x)$ is the modified Bessel function of the first kind.

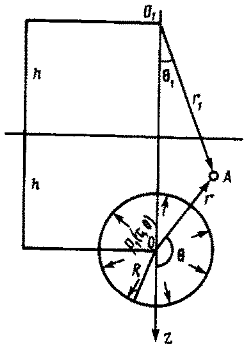

Fig.1

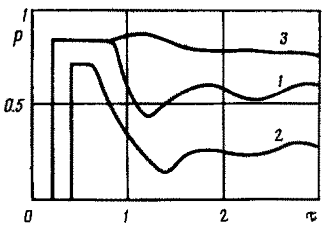

Fig.2

Expanding the transform of the given pressure $p_1^L(s,\theta)$ in series in Legendre polynomials with coefficients $p_{n1}^L(s)$ and substituting (2.4) into the second boundary conditions (1.1), we obtain an infinite system of algebraic equations in $A_n^L(s)$

$$M_n(s) D_n^L(s) e^{-2s} + \sum_{p=0}^{\infty} C_{np}^{(1)}(s) D_p^L(s) e^{-2hs} + \tag{2.5}$$
$$\sum_{p=0}^{\infty} C_{np}^{(2)}(s) D_p^L(s) e^{-2hs} e^{-2s} = -\frac{p_{n1}^L(s)}{s} e^{-s}$$

$$M_n(s) = \frac{R_{n0}(s)}{s^n}, \qquad D_n^L(s) = \sqrt{\frac{\pi}{2s}} A_n^L(s)$$

$$C_{np}^{(1)}(s) = (-1)^{p+1}(2n+1)\frac{R_{n0}(-s)}{4hs^{n+1}} \sum_{\sigma=|p-n|}^{p+n} b_\sigma^{(n0p0)} \frac{R_{\sigma0}(2hs)}{(2hs)^\sigma}$$

$$C_{np}^{(2)}(s) = (-1)^{p}(2n+1)\frac{R_{n0}(s)}{4hs^{n+1}} \sum_{\sigma=|p-n|}^{p+n} b_\sigma^{(n0p0)} \frac{R_{\sigma0}(2hs)}{(2hs)^\sigma}$$

$$R_{m0}(s) = \sum_{k=0}^{m} A_{mk} s^{m-k}, \qquad A_{mk} = \frac{(m+k)!}{(m-k)!\,k!\,2^k}$$

To construct the system of Eqs.(2.5) we took into account the relation between the functions $K_{n+1/2}(x)$ and $I_{n+1/2}(x)$ and elementary functions /3/.

Putting $X = e^{-2hs}$ and $Y = e^{-s}$, we can write the infinite system of algebraic Eqs.(2.5) in the following matrix form:

$$\mathbf{M}DY^2 + \mathbf{C}^{(1)}\mathbf{D}X + \mathbf{C}^{(2)}\mathbf{D}XY^2 = -\mathbf{P}_1 Y \tag{2.6}$$

where $C^{(1)}$ and $C^{(2)}$ are infinite matrices with the elements $C_{np}^{(1)}(s), C_{np}^{(2)}(s)$ respectively, $\mathbf{M}$ is an infinite diagonal matrix with elements $M_n(s)$; $\mathbf{P}_1$ is an infinite vector with elements $p_{n1}^L(s)/s$, and $\mathbf{D}$ is an infinite unknown vector with elements $D_n^L(s)$.

We will represent the solution of the matrix Eq.(2.6) in the form of a series in exponential functions

$$\mathbf{D} = \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} \mathbf{d}_{ij}(s) X^i Y^{-j} \tag{2.7}$$

where $\mathbf{d}_{ij}(s)$ is an infinite vector with elements $d_{ij}^{(n)}(s)$.

Note that for a finite time τ, series (2.7) reduces to a finite sum.

Substituting expression (2.7) into (2.6) and equating coefficients of the same powers of X and Y, we obtain the following recurrence relations:

$$\begin{gathered} \mathbf{Md}_{0,1} = -\mathbf{P}_1, \quad \mathbf{Md}_{i,1} + \mathbf{C}^{(2)}\mathbf{d}_{i-1,1} = 0 \quad (i \geqslant 1) \\ \mathbf{d}_{0,2} = 0, \quad \mathbf{Md}_{i,2} + \mathbf{C}^{(2)}\mathbf{d}_{i-1,2} = 0 \quad (i \geqslant 1) \\ \mathbf{d}_{0,j+2} = 0 \quad (j \geqslant 1), \quad \mathbf{Md}_{i,j+2} + \mathbf{C}^{(1)}\mathbf{d}_{i-1,j} + \mathbf{C}^2\mathbf{d}_{i-1,j+2} = 0 \quad (i \geqslant 1, \ j \geqslant 1) \end{gathered} \tag{2.8}$$

which enables us to determine all the elements $d_{ij}^{(n)}(s)$ of the columns $\mathbf{d}_{ij}(s)$ in the form of rational functions. Then, we obtain the following expression for the pressure in the medium in image space:

$$\begin{gathered} p_n^L(r,s) = \frac{1}{r} \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} [sM_n(sr)\, d_{ij}^{(n)}(s)\, e^{-2hsi} e^{-(r-j)s} + \\ \sum_{p=0}^{\infty} (-1)^{n+1} sM_n(-sr)\, C_{np}(s)\, d_{ij}^{(p)}(s)\, e^{-2hs(i+1)}\, e^{-(r+j)s} + \\ \sum_{p=0}^{\infty} sM_n(sr)\, C_{np}(s)\, d_{ij}^{(p)}(s)\, e^{-2hs(i+1)}\, e^{-(r-j)s} \Big] \\ C_{np}(s) = \frac{(-1)^p (2n+1)}{4hs} \sum_{\sigma=|p-n|}^{p+n} b_\sigma^{(n0p0)} \frac{R_{\sigma 0}(2hs)}{(2hs)^\sigma} \end{gathered} \tag{2.9}$$

The coefficients of the exponential functions in (2.9) are rational functions of the transformation parameter s, which enables us to calculate their originals fairly simply, and consequently, obtain the originals of the coefficients of the corresponding series.

Note that the proposed approach can be extended without an fundamental changes to the analogous problem with a velocity specified on the boundary of the cavity (the second boundary condition in (1.1) is changed), to the problem of the diffraction of plane waves by a cavity, a fixed or moving absolutely rigid sphere, and also to the corresponding problems for an elastic half-space.

To realize the algorithm in practice, we used the method of reduction for the infinite matrix Eq.(2.8). The columns $\mathbf{d}_{ij}$ and the matrices $\mathbf{M}, \mathbf{C}^{(1)}$ and $\mathbf{C}^{(2)}$ are replaced by finite columns and matrices with dimensions of $N \times 1$ and $N \times N$. In expression (2.9), instead of the series with the index p, we used corresponding partial sums.

Example. Consider the case of a pressure uniformly distributed over the surface of the cavity: $p_1(\tau,\theta) = p_0 H(\tau)$, where $H(\tau)$ is the Heaviside function. In Fig.2 we show the pressure p as a function of the time τ for $h = 1.5$ and $p_0 = 1$ at three points: $r = 1.2$ and $\theta = \pi$ (curve *1*), $r = 1.4$ and $\theta = \pi$ (curve *2*) and $r = 1.2$ and $\theta = \pi/2$ (curve *3*).

We retained four terms of the series in Legendre polynomials in the calculations. This gives sufficient accuracy, since the sum obtained for the pressure differs from the accurate value when $\tau \geqslant 1.8$ by approximately 7%.

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